International Journal of Theoretical Physics, Vol. 40, No. 12, December 2001 (© 2001)

Superinflation, Quintessence, and the Avoidance of the Initial Singularity

A. Saa,[1,2,3] E. Gunzig,[1,4] L. Brenig,[1,5] V. Faraoni,[1,6,8] T. M. Rocha Filho,[7] and A. Figueiredo[7]

Received March 10, 2001

We consider the dynamics of a spatially flat universe dominated by a self-interacting nonminimally coupled scalar field. The structure of the phase space and complete phase portraits for the conformal coupling case are given. It is shown that the nonminimal coupling modifies drastically the dynamics of the universe. New cosmological behaviors are identified, including superinflation ($\dot{H} > 0$), avoidance of big bang singularities through classical birth of the universe from empty Minkowski space, and spontaneous entry into and exit from inflation. The relevance of this model to the description of quintessence is discussed.

1. INTRODUCTION

The description of the matter content of the cosmos with a single scalar field is appropriate during important epochs of the history of the universe (Kolb and Turner, 1994). In this article, a dynamical system approach to a self-consistent nonsingular cosmological history is presented in the framework of the classical Einstein equations with a nonminimally coupled scalar field. The complete structure of the phase portrait and of the dynamical behavior is presented for the case of a scalar field conformally coupled to the space-time curvature and with a quartic self-interaction potential. This exhaustive analysis is made possible because of the reduction of the dynamics to a two-dimensional manifold embedded in the

[1] RggR, Université Libre de Bruxelles, Brussels, Belgium.
[2] Dep. Física Fonamental, Universitat de Barcelona, Barcelona, Spain.
[3] Dep. Matemática Aplicada, IMECC-UNICAMP, Campinas, São Paulo, Brazil.
[4] Instituts Internationaux de Chimie et de Physique Solvay, Brussels, Belgium.
[5] Service de Physique Statistique, Université Libre de Bruxelles, Brussels, Belgium.
[6] INFN-Laboratori Nazionali di Frascati, Frascati, Rome, Italy.
[7] Instituto de Física, Universidade de Brasilia, Brasília, Distrito Federal, Brazil.
[8] To whom correspondence should be addressed at Department of Physics, University of Northern British Columbia, 3333 University Way, Prince George, Canada. e-mail: vfaraoni@ulb.ac.be.

0020-7748/01/1200-2295/0 © 2001 Plenum Publishing Corporation

original three-dimensional phase space, a general property shown earlier by some of the authors (Gunzig *et al.*, 2000a–c) for a classical scalar field in a spatially flat universe with arbitrary self-interaction potentials and arbitrary coupling to the curvature. Solutions with special dynamical interest are identified, namely heteroclinic and homoclinic solutions in the reduced two-dimensional phase space, and their relevance to a possible classical birth of the universe from empty space is discussed. We recall that heteroclinic trajectories in a phase space correspond to bounded solutions connecting two different fixed points. Typically, they play the role of separatrices, determining regions of the phase space with qualitative distinct dynamical behaviors. Homoclinic trajectories, on the other hand, correspond to solutions starting and ending at the same fixed point. Their relevance to chaotic motions has been intensively discussed in the literature (Ozorio de Almeida, 1994). Despite the fact that the model presented here has no chaotic behavior, its homoclinic solutions will probably mark candidate regions of the phase space for chaotic motions if a small perturbation to the equations is introduced.

Our model consists of a universe filled with a self-interacting nonminimally coupled scalar field. A crucial ingredient of the physics of scalar fields in curved spaces is their nonminimal coupling to the scalar curvature R of space-time, which is required by first loop corrections (Birrell and Davies, 1980; Ford and Toms, 1982; Nelson and Panangaden, 1982; Parker and Toms, 1985), by specific particle theories (Faraoni, 2000), and by scale-invariance arguments at the classical level (Callan *et al.*, 1970). It is well known that nonminimal coupling dictates the success or failure of inflationary models (Abbott, 1981; Futamase and Maeda, 1989); more generally, it turns out to strongly affect the cosmic dynamics, which is qualitatively richer than in the minimally coupled case. We show, indeed, that nonminimal coupling leads to new dynamical behaviors, such as a regime that we propose to call *superinflation* ($\dot{H} > 0$), which cannot be achieved with minimal coupling (Liddle *et al.*, 1994), and spontaneous entry into and exit from inflation, with or without a cosmological constant. Spontaneous superinflation provides a classical alternative to semiclassical birth of the universe from empty Minkowski space (Gunzig *et al.*, 2000a–c; Gunzig and Nardone, 1987), which is impossible with minimal coupling.

In the next section, we review our model, recently proposed in Gunzig *et al.* (in press), and give some definitions. The aspect of the phase portraits are presented in Section 3. Section 4 is devoted to the asymptotic analyses of some special solutions. The last section contains the concluding remarks.

2. THE MODEL

We consider the nonminimally coupled theory described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(-\frac{R}{\kappa} + g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - 2V + \xi R \psi^2 \right), \qquad (1)$$

where R denotes the scalar curvature, ψ is the scalar field, $\kappa \equiv 8\pi G$ (G being Newton's constant), and ξ is the nonminimal coupling constant. A cosmological constant Λ, if present, is incorporated in the scalar field potential $V(\psi)$. We use the fully conserved scalar field stress–energy tensor

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \xi(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(\psi^2) + \xi G_{\mu\nu}\psi^2 - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\psi\partial^\alpha\psi - 2V(\psi)) \tag{2}$$

(where $G_{\mu\nu}$ is the Einstein tensor), thereby avoiding the widespread effective coupling $\kappa_{\text{eff}} = \kappa(1 - \kappa\xi\psi^2)^{-1}$ in the Einstein equations $G_{\mu\nu} = \kappa T_{\mu\nu}$. We consider here the dynamics of a spatially flat Friedmann–Robertson–Walker universe with line element $ds^2 = d\tau^2 - a^2(\tau)(dx^2 + dy^2 + dz^2)$. This yields the trace equation $R = -\kappa(\sigma - 3p)$, the energy constraint $3H^2 = \kappa\sigma$ (which guarantees that the energy density $\sigma \geq 0$), and the Klein–Gordon equation. More explicitly,

$$6[1 - \xi(1 - 6\xi)\kappa\psi^2](\dot{H} + 2H^2) - \kappa(6\xi - 1)\dot{\psi}^2 - 4\kappa V + 6\kappa\xi\psi\,\frac{dV}{d\psi} = 0, \tag{3}$$

$$\frac{\kappa}{2}\dot{\psi}^2 + 6\xi\kappa H\psi\dot{\psi} - 3H^2(1 - \kappa\xi\psi^2) + \kappa V = 0, \tag{4}$$

$$\ddot{\psi} + 3H\dot{\psi} + \xi R\psi + \frac{dV}{d\psi} = 0. \tag{5}$$

The (time-dependent) equation of state of the ψ field, rather than being imposed a priori, follows self-consistently from the dynamics. In the trace equation (3), the second derivative $\ddot{\psi}$ that appears in the pressure has been replaced by its expression given by the Klein–Gordon equation. Clearly, in the set of Eqs. (3)–(5), the subsystem [(3) and (4)] is a closed implicit two-dimensional system for ψ and $H = \dot{a}/a$ (note that this dimensional reduction is not possible for spatially curved universes (Amendola *et al.*, 1990; Gunzig *et al.*, 2000a–c). After solving these implicit equations for $\dot{\psi}$ and $\dot{H}$, one has

$$\dot{\psi} = -6\xi H\psi \pm \frac{1}{2\kappa}\sqrt{\mathcal{G}(H, \psi)} \tag{6}$$

$$\begin{aligned}\dot{H} = {} & \frac{1}{1 + \kappa\xi(6\xi - 1)\psi^2}[3(2\xi - 1)H^2 + 3\xi(6\xi - 1)(4\xi - 1)\kappa H^2\psi^2 \\ & \mp \xi(6\xi - 1)H\psi\sqrt{\mathcal{G}} + (1 - 2\xi)\kappa V(\psi)],\end{aligned} \tag{7}$$

where

$$\mathcal{G}(H, \psi) = 8\kappa^2\left[\frac{3H^2}{\kappa} - V(\psi) + 3\xi(6\xi - 1)H^2\psi^2\right]. \tag{8}$$

Because of the energy constraint (4), the trajectories are restricted to a two-dimensional manifold Σ in the three-dimensional $(H, \psi, \dot{\psi})$ original phase space, possibly with "holes" (dynamically forbidden regions) corresponding to

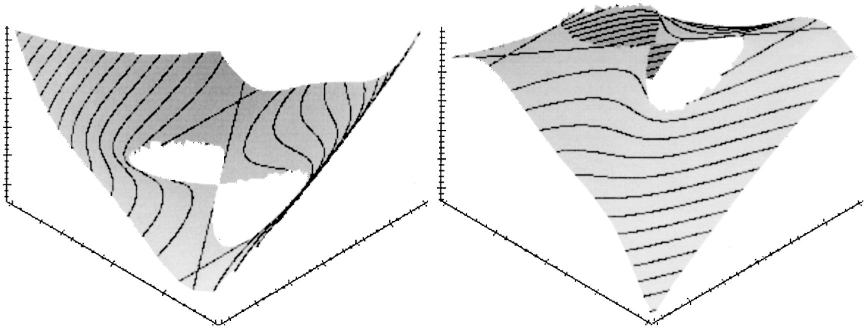

Fig. 1. Aspect of the two-dimensional manifold Σ, embedded in the three-dimensional phase space $(\psi, H, \dot{\psi})$, for the potential (9). The upper graph represents the "+" sheet, while the lower one depicts the "−" sheet. In fact, they are not disconnected; the two sheets join smoothly on the boundary $\mathcal{G} = 0$ of the dynamically forbidden region, corresponding to the shown "holes." The lines of constant $\mathcal{G}$ are drawn on the sheets.

$\mathcal{G}(H, \psi) < 0$ (cf. Eq. (6)). Σ is composed of two sheets corresponding to the positive or negative sign in Eq. (6). The two sheets smoothly join on the boundary $\mathcal{G} = 0$ of the dynamically forbidden region. In the present work, we restrict to the potential

$$V(\psi) = \frac{3\alpha}{\kappa}\psi^2 - \frac{\Omega}{4}\psi^4 - \frac{9\omega}{\kappa^2}, \tag{9}$$

consisting of a mass term, a quartic self-coupling, and possibly, a cosmological constant term. For consistency with previous works (Gunzig *et al.*, 2000a–c) we use the symbols $\alpha \equiv \kappa m^2/6$ (m being the scalar field mass) and $\omega \equiv -\kappa^2\Lambda/9$. Figures 1 and 2 present some aspects of Σ for the potential (9). However, several of our main results do not depend on the details of $V(\psi)$.

3. PHASE SPACE PORTRAITS

In the following, for simplicity, we project the dynamics of the phase space onto the (H, ψ) plane, but the true nature of Σ should always be kept in mind. The fixed points of the system [(3)–(5)] include de Sitter solutions with constant scalar field

$$H_0^2 = \frac{3(\alpha^2 - \Omega\omega)}{\kappa(\Omega - 6\xi\alpha)}, \qquad \psi_0^2 = \frac{6(\alpha - 6\xi\omega)}{\kappa(\Omega - 6\xi\alpha)} \tag{10}$$

$(\Omega \neq 6\alpha\xi)$, and the solutions $(H, \psi) = (\pm\sqrt{-3\omega/\kappa}, 0)$. The fixed points (10) exist also for $\omega = 0$ because of the presence of the matter field ψ (in this case, the two points $(\pm\sqrt{-3\omega/\kappa}, 0)$ collapse into the Minkowski space fixed point, which

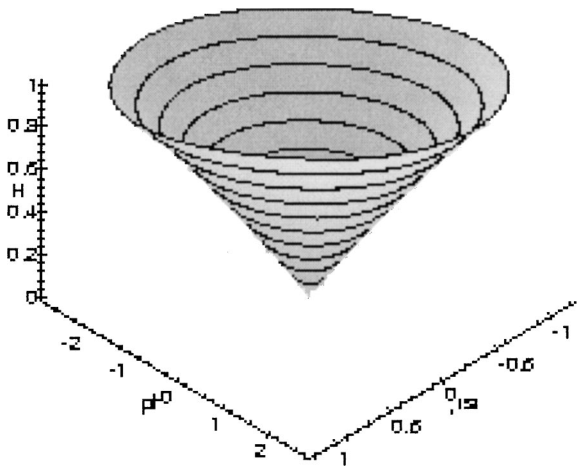

Fig. 2. Zoom, near the origin, of the two-dimensional manifold Σ, embedded in the three-dimensional phase space $(\psi, \dot{\psi}, H)$, for the potential (9). The two sheets join, forming two symmetric cones with their apex at the origin. Only the cone corresponding to $H > 0$ is presented here, and the lines of constant H are drawn.

is at the apex of the cones described in Fig. 2). Here, we restrict ourselves to the case of conformal coupling, $\xi = 1/6$.

The function

$$L(\psi, \dot{\psi}) = \frac{1}{2}\dot{\psi}^2 + \frac{\alpha}{4}\psi^4 - \frac{3\omega}{\kappa}\psi^2 + V(\psi) \tag{11}$$

is such that $dL/dt = -3H\dot{\psi}^2$ along the trajectories. For $H > 0$, L is a Lyapunov function in a region containing the origin; the solutions are then confined by closed lines of constant L, implying asymptotic convergence to the fixed points on the H axis (from Eq. (4), if ψ and $\dot{\psi}$ vanish, H goes to $V(0)$). This behavior is confirmed by exhaustive numerical simulations (Gunzig *et al.*, in press) and reported in the following. We first exclude a cosmological constant by setting $\omega = 0$. The phase portrait qualitatively differs according to the ratio Ω/α.

3.1. The Case $\Omega = 2\alpha$

The Minkowski space $(H, \psi, \dot{\psi}) = (0, 0, 0)$ is a fixed point, attractive for $H > 0$ and repulsive for $H < 0$; the projections of the de Sitter spaces $(\pm H_0, \pm\psi_0, 0)$ are saddle points, i.e., they possess attractive and repulsive eigendirections in the phase space (Fig. 3(a)). They are of two kinds: expanding $(H\psi > 0)$ or

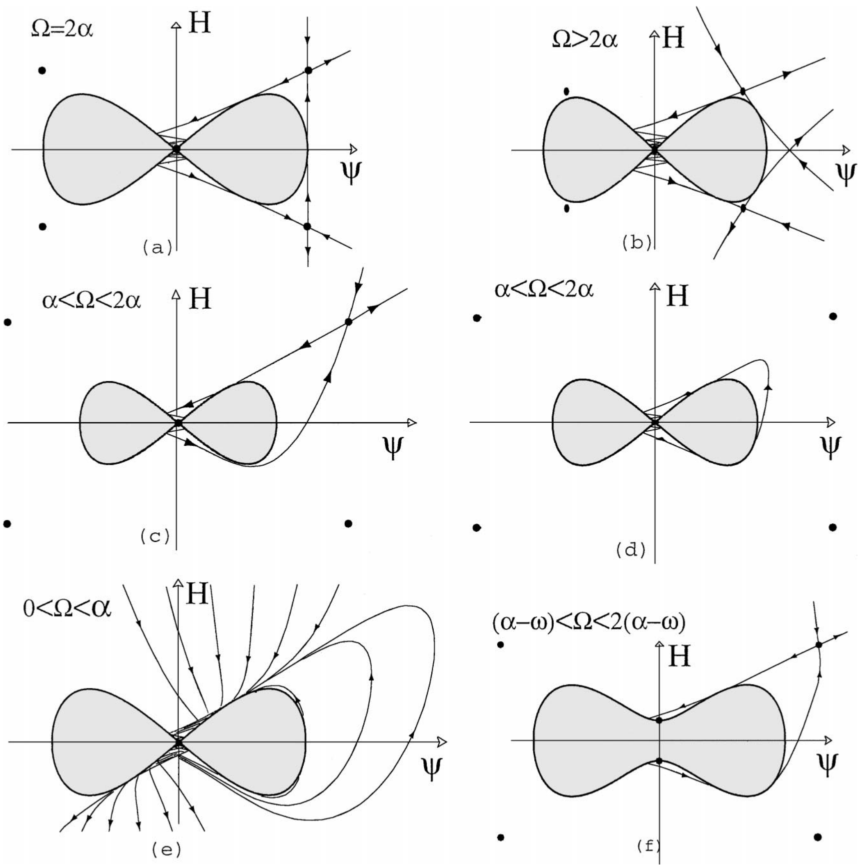

Fig. 3. Qualitative phase portraits for the system [(3)–(5)]. Shadowed regions correspond to the dynamically forbidden regions ($\mathcal{G}(H, \psi) < 0$, cf. Eq. (6)). (a)–(e) were obtained by using $\Omega/\alpha = 2, 5, 3/2, 3/2$, and $1/2$, and respectively, $\omega = 0$; (f) corresponds to the case $\omega = 1/10$ and $\Omega/(\alpha - \omega) = 3/2$.

contracting ($H\psi < 0$). The following solutions, present only in this particular case,

$$H(\tau) = \sqrt{\frac{C}{2}}\,\tanh(\sqrt{2C}\tau), \qquad \psi = \pm\psi_0 \equiv \pm\sqrt{\frac{6}{\kappa}} \tag{12}$$

(where $C = \dot{H} + 2H^2 = -R/6$ is constant) correspond to heteroclinic straight lines connecting de Sitter fixed points, starting along the repulsive eigendirection of one of them and ending along the attractive eigendirection of the other (Fig. 3(a)). They are tangent to the boundary of the forbidden regions at $(H, \psi) = (0, \pm\psi_0)$.

For $|H| > \sqrt{C/2}$, another straight-line solution is obtained from the general form

$$H(\tau) = \sqrt{\frac{C}{2}} \frac{w_1 e^{\sqrt{C/2}\tau} - w_2 e^{-\sqrt{C/2}\tau}}{w_1 e^{\sqrt{C/2}\tau} + w_2 e^{-\sqrt{C/2}\tau}}, \qquad \psi = \psi_0, \tag{13}$$

where w_1 and w_2 are integration constants. The nonsingular solutions (12) connect a contracting ($\tau \to -\infty$) de Sitter regime to a minimum nonvanishing value of the scale factor ($\tau = 0$), and then to an expanding de Sitter regime ($\tau \to +\infty$).

In addition to these straight lines, we found numerically other heteroclinic solutions: one starting at $(H, \psi) = (0, 0)$ and ending at $(-H_0, \psi_0)$, and another one from (H_0, ψ_0) to $(0, 0)$. A third solution starts at the expanding de Sitter point and goes to infinity, while another one comes from infinity and arrives to the contracting de Sitter point. The phase portrait is symmetric about the origin.

Near the fixed point $(0, 0, 0)$, numerical analysis confirms the peculiar behavior suggested by the Lyapunov function: orbits approaching this point with positive H are attracted to it, bouncing back and forth infinitely many times off the $\mathcal{G} = 0$ boundary in the (H, ψ) projection (Fig. 3(a)). In the space $(H, \psi, \dot{\psi})$ these orbits are seen to spiral down on a cone toward its apex at the origin. The cone results from the union of the two sheets in the vicinity of the origin, as it is shown in Fig. 3. Along the spiral, the orbit passes almost periodically from one sheet to the other, with period $\tau_{\text{bounce}} = 2\pi/m$ (τ_{bounce} is obtained from the asymptotic analysis of the next section. Typically, after a few bounces, the period coincides with τ_{bounce}, with good accuracy). A similar behavior for $\Omega < 0$ was reported in the earlier numerical analysis of Amendola *et al.* (1990), but using the effective coupling $\kappa_{\text{eff}}(\tau)$ and the variables ψ and $\dot{\psi}$.

In the $H < 0$ half-plane, the situation is reversed: orbits starting with $H < 0$ are repelled by the origin and depart from it bouncing off the $\mathcal{G} = 0$ boundary.

3.2. The Case $\Omega > 2\alpha$

The situation (Fig. 3(b)) is analogous to the previous one, but now the straight heteroclinic solutions are missing, and are replaced by the solution starting at the contracting de Sitter fixed point and escaping to infinity, and by the solution coming from infinity and arriving to the expanding de Sitter point. The two-sheeted structure of Σ implies that no actual intersections occur between different orbits in Fig. 3(b), which live in different sheets but are projected on the same plane.

3.3. The Case $\alpha < \Omega < 2\alpha$

As shown in Fig. 3(c), there are no straight heteroclinic lines but new interesting features emerge. A new heteroclinic solution appears starting from the origin and ending in the expanding de Sitter fixed point. As in the previous case, the quadrant ($\psi > 0$, $H < 0$) is obtained from the ($\psi > 0$, $H > 0$) one by reflection about the ψ-axis and time-reversal in Fig. 3(c).

The crucial feature of this case is the appearance of a dense set of homoclinic solutions (Fig. 3(d)) departing from the origin with negative H and returning to it with positive H, going around the forbidden region. Superinflation plays a central role along these orbits; only a regime with $\dot{H} > 0$ permits the smooth transition from an initial contracting ($H < 0$) phase to an expanding one ($H > 0$). This transition occurs at the nonvanishing minimum of the scale factor. The behavior of this family of homoclinics, as well as of the other solutions, is universal: they rapidly converge in the spiraling region near the origin, irrespective of initial conditions. Since all these homoclinics originate from a neighborhood of the Minkowski fixed point because of its own instability with respect to perturbations with $H < 0$, and come back to that point because of the stability for perturbations with $H > 0$, this behavior constitute a classical alternative to the previously proposed semiclassical birth of the universe from empty space (Gunzig *et al.*, 2000a–c; Gunzig and Nardone, 1997).

3.4. The Case $0 < \Omega < \alpha$

The fixed points (10) disappear and the only bounded solutions are the homoclinics associated with the origin (see Fig. 3(e)). This situation is therefore the most favorable for the classical spontaneous exit from empty Minkowski space.

3.5. The Case $\omega < 0$

Analogous results hold if a small cosmological constant is present (see Fig. 3(f)), with the phase portrait being classified according to $\Omega/(\alpha - \omega)$, but the fixed point (0, 0, 0) of the $\omega = 0$ case splits into two de Sitter fixed points with memory of the previous stability properties. Now, the approximate period between two consecutive bounces is

$$\tau_{\text{bounce}} = \frac{2\pi}{\sqrt{m^2 + \frac{3\omega}{4\kappa}}}. \tag{14}$$

4. ASYMPTOTIC ANALYSIS

For the conformally coupled case, the trace and the Klein–Gordon equations (3) and (5), after a rescaling and the explicit substitution of the Hamiltonian constraint (4), read

$$\begin{aligned} \dot{H} + 2H^2 - \alpha\psi^2 + 6\omega = 0 \\ \ddot{\psi} + 3H\dot{\psi} + 6(\alpha - \omega)\psi - (\Omega - \alpha)\psi^3 = 0. \end{aligned} \tag{15}$$

We have special interest in the asymptotic behavior of the solutions of (15) in the vicinity of the attractive fixed points. Let us start with the case $\omega = 0$. In this case, our region of interest is the neighborhood of the origin with $H > 0$. We search for asymptotic solutions of the form

$$\begin{aligned} \psi(t) &= \frac{f_1(t)}{t} + \frac{f_2(t)}{t^2} + \mathcal{O}(t^{-3}) \\ H(t) &= \frac{g_1(t)}{t} + \frac{g_2(t)}{t^2} + \mathcal{O}(t^{-3}), \end{aligned} \tag{16}$$

with $f_1(t)$, $f_2(t)$, $g_1(t)$, and $g_2(t)$ bounded for large t. Inserting (16) in the Eqs. (15) one has

$$\begin{aligned} \frac{1}{t}(f_1'' + 6\alpha f_1) + \frac{1}{t^2}(f_2'' - 2f_1' + 3g_1 f_1' + 6\alpha f_2) &= \mathcal{O}(t^{-3}) \\ \frac{1}{t}g_1' + \frac{1}{t^2}\left(g_2' - g_1 + 2g_1^2 - \alpha f_1^2\right) &= \mathcal{O}(t^{-3}). \end{aligned} \tag{17}$$

By demanding the exactness of (17) up to t^{-3} order, one gets immediately that g_1 is a constant and $f_1 = A\cos(\sqrt{6\alpha}t + \delta)$, leading to

$$f_2'' + 6\alpha f_2 = A\sqrt{6\alpha}(3g_1 - 2)\sin(\sqrt{6\alpha}t + \delta), \tag{18}$$

which has the general solution

$$\begin{aligned} f_2 = B\cos(\sqrt{6\alpha}t + \theta) &- \frac{A(3g_1 - 2)}{2\sqrt{6\alpha}} \\ &\times (\cos(\sqrt{6\alpha}t + \delta)\sqrt{6\alpha}t - \cos(\sqrt{6\alpha}t)\sin\delta). \end{aligned} \tag{19}$$

In order to guarantee the boundedness of f_2, it is necessary to have $g_1 = 2/3$. From the equation corresponding to the t^{-2} term in the trace equation, we have

$$g_2' = A^2\alpha\cos^2(\sqrt{6\alpha}t + \delta) - \frac{2}{9}, \tag{20}$$

which is solved by

$$g_2 = \frac{A^2\alpha}{2\sqrt{6\alpha}}(\cos(\sqrt{6\alpha}t + \delta)\sin(\sqrt{6\alpha}t + \delta) + \delta) + \left(\frac{A^2\alpha}{2} - \frac{2}{9}\right)t. \tag{21}$$

Again, by requiring the boundedness of g_2 for large t, we get the condition $A = 2/(3\sqrt{\alpha})$. We have, finally, the following asymptotic solution for the $\omega = 0$ case:

$$\begin{aligned} \psi(t) &= \frac{2\cos\sqrt{6\alpha}t}{3\sqrt{\alpha}t} + \mathcal{O}(t^{-2}) \\ H(t) &= \frac{2}{3t} + \mathcal{O}(t^{-2}). \end{aligned} \tag{22}$$

From (22) we can obtain the characteristic period $\tau_{\text{bounce}} = 2\pi/\sqrt{6\alpha}$. Also, the asymptotic solution (22) implies that all solutions ending in the origin will behave as matter-dominated universes for large t, corresponding to $a(t) \propto t^{2/3}$.

The case with a small cosmological constant can be treated analogously, taking into account that now the relevant fixed point is $\psi = 0$ and $H = \sqrt{-3\omega}$. The search for asymptotic solutions of the form

$$\begin{aligned} \psi(t) &= \frac{f_1(t)}{t} + \frac{f_2(t)}{t^2} + \mathcal{O}(t^{-3}) \\ H(t) &= g_0(t) + \frac{g_1(t)}{t} + \frac{g_2(t)}{t^2} + \mathcal{O}(t^{-3}), \end{aligned} \tag{23}$$

with $f_1(t)$, $f_2(t)$, $g_0(t)$, $g_1(t)$, and $g_2(t)$ bounded for large t, as before, give rise to the equations

$$\begin{aligned} &\frac{1}{t}(f_1'' + 3g_0 f_1' + 6(\alpha - \omega) f_1) + \frac{1}{t^2}(f_2'' - 2 f_1' \\ &\quad + 3(g_1 f_1' + g_0 f_2' - g_0 f_1) + 6(\alpha - \omega) f_2) = \mathcal{O}(t^{-3}) \\ &(g_0' + 2g_0 + 6\omega) + \frac{1}{t}(g_1' + 4g_0 g_1) + \frac{1}{t^2} \\ &\quad \times \left(g_2' - g_1 + 4g_0 g_1 + 2g_1^2 - \alpha f_1^2\right) = \mathcal{O}(t^{-3}). \end{aligned} \tag{24}$$

Our asymptotic equations are now considerably more complicated, but nevertheless, we can obtain an unambiguous period τ_{bounce}. From (24), one has that g_0 decreases exponentially to the value $\sqrt{-3\omega}$ for large t, implying that f_1 converges to the form $f_1 = A\exp(-3\sqrt{-3\omega}t/2)\cos(\sqrt{6\alpha + 3\omega/4}t + \delta)$ from which one can obtain $\tau_{\text{bounce}} = 2\pi/\sqrt{6\alpha + 3\omega/4}$. The scale factor, in this case, obeys $a(t) \propto \exp\sqrt{-3\omega}t$ for large t, as in the de Sitter universe.

5. DISCUSSIONS

The (H, ψ) plane is divided into sectors by the straight lines $H = \pm\sqrt{\alpha/2}\psi$, $H = \pm\sqrt{\alpha}\psi$, and $H = \pm\sqrt{2\alpha}\psi$ corresponding, respectively, to $\dot{H} = 0$, $\ddot{a} = 0$, and pressure $p = 0$ (the H-axis corresponds to $p = \sigma/3$). The lines $H = \pm\sqrt{\alpha/2}\psi$ mark the transition between inflationary ($\ddot{a} > 0$ and $\dot{H} \le 0$ and superinflationary ($\dot{H} > 0$) regimes. The lines $H = \pm\sqrt{\alpha}\psi$ divide regions corresponding to inflation and to decelerated expansion, while $H = \pm\sqrt{2\alpha}\psi$ divide regions of positive and negative pressures (Fig. 4). The crucial condition for superinflation to occur is that the line $\dot{H} = 0$ (or parts of it) should belong to the dynamically accessible region $\mathcal{G} \ge 0$ of the (H, ψ) plane. This implies that, for an arbitrary potential $V(\psi)$, superinflation corresponds to $\psi\, dV/d\psi \le 0$ (this result will be discussed in detail elsewhere). For our particular potential (9), this requires $\Omega > 0$. The

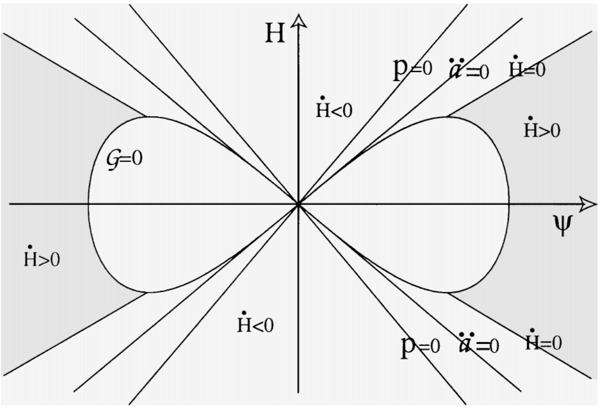

Fig. 4. The plane (H, ψ) for the $\omega = 0$ case and the equation of state for ψ. The darkest region corresponds to the superinflation regime ($\dot{H} > 0$). During the bounces of the ψ solution in the region $\dot{H} < 0$, its equation of state corresponds to, respectively, radiation domination (crossing the H-axis), matter domination ($p = 0$), and reacceleration (between $\mathcal{G} = 0$ and $\ddot{a} = 0$ lines). These oscillations are damped as $\tau \to +\infty$; the universe becomes matter dominated (i.e., $a(\tau) \propto \tau^{2/3}$ and tends to infinite dilution).

superinflationary behavior occurs only once along each homoclinic and brings the solution from the primordial Minkowski neighborhood to the succession of eras corresponding to different equations of state (during each bounce), toward infinite dilution and equation of state $p = 0$. In fact, asymptotic analysis for $\tau \to +\infty$ and for any value of α and Ω (with $\omega = 0$) shows that the scale factor $a(\tau)$ exhibits oscillations of concavity corresponding to accelerated and decelerated epochs. These oscillations are damped as $\tau \to \infty$; in this regime, $a(\tau) \propto \tau^{2/3}$ and the universe becomes matter dominated.

While it is not claimed here that the evolution of our universe is modeled by an entire orbit of the system [(3)–(5)] on the accessible manifold Σ, the application to specific eras of the cosmological history is intriguing. Indeed, in the bounces reported above (during which $\dot{H} < 0$), one encounts, respectively, radiation domination crossing the H-axis, matter domination ($p = 0$), acceleration (a possible quintessence model?) until the next bounce in the (H, ψ) projection, where this sequence is reversed.

If we identify one period as our cosmological history, then the reported accelerated expansion of the universe today (Perlmutter *et al.*, 1998, 1999; Riess *et al.*, 1998) suggests to locate our epoch in the sector between the line $H = \sqrt{\alpha}\psi$ and the $\mathcal{G} = 0$ boundary. The identification of the age of the universe ($\sim 10^{17}$ s) with τ_{bounce} would then yield the scalar field mass $m \simeq 10^{-13}$ eV, which is suggestive of an axion (Kolb and Turner, 1994) or of an ultralight pseudo-Goldstone boson (that the quintessence field should be very light was already suggested (Binetruy, 2000).

As a step toward a realistic model, one can include a second scalar field coupled to ψ, which has the meaning of a fundamental field (as is done, e.g., in hybrid inflation), or mimics a baryonic or other fluid. In spite of the higher dimensionality of the phase space, many of the features exposed here for a single scalar field survive (Gunzig *et al.*, work in progress). Although very simple, we think our classical model opens interesting avenues to the understanding of quintessence in terms of a nonminimally coupled scalar field.

Finally, one should keep in mind that our detailed semianalytical analyses are possible because of the reduction of the original three-dimensional system to a two-dimensional one on a smooth manifold Σ, which, in turn, strongly indicates the absence of chaos (confirmed by an exhaustive numerical analysis). Generically, such a reduction is not possible for the case of a spatially curved space-time and/or several matter fields nonminimally coupled to the curvature, despite of many similar results. This is the object of present investigations (Gunzig *et al.*, work is progress).

ACKNOWLEDGMENTS

We acknowledge the Centre de Calcul Symbolique sur Ordinateur for the use of computer facilities, and financial support from the EEC (Grant No. HPHA-CT-2000-00015), from OLAM, Fondation pour la Recherche Fondamentale (Brussels), from FAPESP (São Paulo, Brazil), and from the ESPRIT Working Group CATHODE.

REFERENCES

Abbott, L. F. (1981). *Nuclear Physics B* **185**, 233.

Amendola, L., Litterio M., and Occhionero, F. (1990). *International Journal of Modern Physics A* **5**, 3861.

Binetruy, P. (2000). *International Journal of Theoretical Physics* **39**, 1859.

Birrell, N. D. and Davies, P. C. (1980). *Physical Review D: Particles and Fields* **22**, 322.

Callan, C. G., Jr, Coleman, S., and Jackiw, R. (1970). *Annals of Physics* (New York) **59**, 42.

Faraoni, V. (1996). *Physical Review D: Particles and Fields* **53**, 6813.

Faraoni, V. (2000). Preprint hep-th/0009053.

Faraoni, V. (2001). A crucial ingredient of inflation. *International Journal of Theoretical Physics* **40**, 2259. Preprint hep-th/0009053.

Ford, L. H. and Toms, D. J. (1982). *Physical Review D: Particles and Fields* **25**, 1510.

Futamase, T. and Maeda, K. (1989). *Physical Review D: Particles and Fields* **39**, 399.

Gunzig, E. and Nardone, P. (1987). *Fund. Cosm. Phys.* **11**, 311 and references therein.

Gunzig, E., Brenig, L., Figueiredo, A., and Rocha Filho, T. M. (2000b). A no-chaos theorem for nonminimally coupled scalar field cosmology and a new cosmogenesis scenario. *Modern Physics Letters A* **15**, 1363.

Gunzig, E., Faraoni, V., Rocha Filho, T. M., Figueiredo, A., and Brenig, L. (2000a). The dynamical systems approach to scalar field cosmology. *Classical and Quantum Gravity* **17**, 1783.

Gunzig, E., Faraoni, V., Rocha Filho, T. M., Figueiredo, A., and Brenig, L. (2000c). What can we learn from nonminimally coupled scalar field cosmology? *International Journal of Theoretical Physics* **39**, 1901.

Gunzig, E., Saa, A., Brenig, L., Faraoni, V., Rocha Filho, T. M., and Figueiredo, A. (2001). Superinflation, quintessence, and nonsingular cosmologies. *Physical Review D: Particles and Fields* **63**, 067301. gr-qc/0012085.

Kolb, E. W and Turner, M. S. (1994). *The Early Universe*, Addison-Wesley, Mass.

Liddle, A. A., Parsons, P., and Barrow, J. D. (1994). *Physical Review D: Particles and Fields* **50**, 7222.

Nelson, B. and Panangaden P. (1982). *Physical Review D: Particles and Fields* **25**, 1019.

Ozorio de Almeida, A. M. (1994). *Hamiltonian Systems: Chaos and Quantization*, Cambridge University Press, Cambridge, U.K.

Parker, L. and Toms, D. J. (1985). *Physical Review D: Particles and Fields* **28**, 1584.

Perlmutter, S. *et al.* (1998). *Nature* **391**, 51.

Perlmutter, S. (1999). *Physical Review Letters* **83**, 670.

Riess, A. G. *et al.* (1998). *Astrophysical Journal* **116**, 1009.

Sonego, S. and Faraoni, V. (1993). *Classical and Quantum Gravity* **10**, 1185.